

Publicacions Matemàtiques, Vol. **44** (2000), 491–501**SOLVABLE GROUPS WITH MANY *BFC*-SUBGROUPS**

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Abstract

We characterize the solvable groups without infinite properly ascending chains of non-*BFC* subgroups and prove that a non-*BFC* group with a descending chain whose factors are finite or abelian is a Černikov group or has an infinite properly descending chain of non-*BFC* subgroups.

0. Introduction

In a series of papers Belyaev-Sesekin [2], Belyaev [3], Bruno-Phillips [4], [5], Kuzucuoğlu-Phillips [12], Leinen-Puglisi [13], Asar [1], Leinen [14] have obtained the results on minimal non-*FC* groups. In particular, in [2] are characterized the minimal non-*BFC* groups, i.e. the non-*BFC* groups in which every proper subgroup is *BFC*. Recall that a group G is called a *BFC*-group if there is a positive integer d such that no element of G has more than d conjugates. Due to the well known result of B. H. Neumann (see e.g. [16, Theorem 4.35]) the *BFC*-groups are precisely the groups with the finite commutator subgroups.

We say that a group G satisfies the minimal condition on non-*BFC* subgroups (for short $\text{Min-}\overline{BFC}$) if for every properly descending series $\{G_n \mid n \in \mathbb{N}\}$ of subgroups of G there exists a number $n_0 \in \mathbb{N}$ such that G_n is a *BFC*-group for every integer $n \geq n_0$ and a group G satisfies maximal condition on non-*BFC* subgroups (for short $\text{Max-}\overline{BFC}$) if there exists no infinite properly ascending series of non-*BFC* subgroups in G . Every minimal non-*BFC* group satisfies $\text{Min-}\overline{BFC}$ and $\text{Max-}\overline{BFC}$. S. Franciosi, F. de Giovanni and Ya. P. Sysak [11] have investigated the locally graded groups with the minimal condition on non-*FC* subgroups.

In this paper we characterize the solvable groups satisfying $\text{Max-}\overline{BFC}$ and $\text{Min-}\overline{BFC}$, respectively. Namely, we prove the two following theorems.

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Key words. *BFC*-group, minimal non-*BFC* group, maximal condition, minimal condition, solvable group.

Theorem 1. *A solvable group G satisfies $\text{Max-}\overline{BFC}$ if and only if it is of one of the following types:*

- (i) G is a BFC -group;
- (ii) $G = BU$ is a finitely generated group, where B is a proper torsion normal subgroup of G , U its polycyclic subgroup and $B\langle x \rangle$ is either a BFC -subgroup or a finitely generated subgroup for every element x of U ;
- (iii) $G = DU$ is a locally nilpotent-by-finite group with the torsion commutator subgroup G' , where D is a normal divisible abelian p -subgroup, U is a polycyclic subgroup, and if $\langle u \rangle$ acts non-trivially on D for an element u of U , then D is an indecomposable injective $\mathbb{Q}\langle u \rangle$ -module and $A\langle u \rangle$ is a BFC -subgroup for every proper submodule A of a $\mathbb{Z}\langle u \rangle$ -module D with the action induced by the conjugation of u on D .

Theorem 2. *Let the group G have a descending series whose factors are finite or abelian. If G satisfies the minimal condition on non- BFC subgroups, then it is a BFC -group or a Černikov group.*

Throughout this paper p is a prime. For a group G , $Z(G)$ will always denote the centre of G , $G', G'', \dots, G^{(n)}$ the terms of derived series of G , $\tau(G)$ the set of all torsion elements of G , $G^p = \langle g^p \mid g \in G \rangle$. In the sequel we will use the following notation:

\mathbb{Q} the rational number field;

\mathbb{F}_p the finite field with p elements;

\mathbb{Q}_p the additive group of all rational numbers whose denominators are p -numbers;

\mathbb{Z} the additive group of all rational integers;

\mathbb{C}_{p^∞} the quasicyclic p -group;

$R\langle x \rangle$ the group ring of a cyclic group $\langle x \rangle$ over a commutative ring R .

We will also use other standard terminology from [10] and [16].

1. Solvable groups with $\text{Max-}\overline{BFC}$

In this section we study the solvable groups with the maximal condition on non- BFC subgroups.

Lemma 1.1. *Let G be a group satisfying $\text{Max-}\overline{BFC}$ and H its subgroup. Then:*

- (i) *H satisfies $\text{Max-}\overline{BFC}$;*
- (ii) *if H is normal in G , then the quotient group G/H satisfies $\text{Max-}\overline{BFC}$;*
- (iii) *if H is a normal non- BFC subgroup of G , then G/H satisfies the maximal condition on subgroups.*

Proof: Is immediate. \square

Lemma 1.2. *Let G be a group which satisfies $\text{Max-}\overline{BFC}$. If G contains a normal abelian subgroup N with the quasicyclic quotient group G/N , then G is a nilpotent group.*

Proof: We prove this lemma by the same arguments as in the proof of Lemma 2.3 from [2]. Since G/N is a quasicyclic p -group for some prime p ,

$$G/N = \bigcup_{n=1}^{\infty} \langle \bar{a}_n \rangle,$$

where $\bar{a}_n^p = \bar{a}_{n-1}$, $\bar{a}_0 = N$. Put $A_n = \langle N, a_n \rangle$. Then $A_n \triangleleft G$, $A_n' \triangleleft G$ and by Lemma 1.1(iii) A_n is a BFC -subgroup. Hence $A_n' \leq Z(G)$ and consequently

$$G' = \bigcup_{n=1}^{\infty} A_n' \leq Z(G),$$

as desired. \square

Lemma 1.3. *If G is a Černikov group with $\text{Max-}\overline{BFC}$, then it is a BFC -group or the quotient group G/G' is finite.*

Proof: Assume that the quotient group $\overline{G} = G/G'$ is infinite and G is not a BFC -group. Then by Theorem 21.3 of [10] $\overline{G} = \overline{D} \times \overline{F}$ is a direct product of the non-trivial divisible part \overline{D} and a reducible subgroup \overline{F} . Let D and F be the inverse images of \overline{D} and \overline{F} in G , respectively. By Corollary 2.2 of [2] $G' = D'F'$. Since G is not a BFC -group, \overline{F} is a finite group. It is clear that $\overline{D} \cong \mathbb{C}_{p^\infty}$ for some prime p and G has a normal BFC -subgroup N with $G/N \cong \mathbb{C}_{p^\infty}$. By Theorem 1.16 of [7] $G = NZ(G)$ and so $G' = N'$, a contradiction with our assumption. The lemma is proved. \square

Proposition 1.4. *If a group G satisfies $\text{Max-}\overline{BFC}$, then it is a BFC -group or the quotient group G/G' is finitely generated.*

Proof: As it is well known $\overline{G} = G/G' = \overline{D} \times \overline{S}$ is a direct product of the divisible part $\overline{D} = D/G'$ and a reducible subgroup $\overline{S} = S/G'$.

(1) First, let \overline{D} be a non-trivial subgroup. Then S and G' are the *BFC*-subgroups. It is clear that G is a *BFC*-group or \overline{D} is a quasicyclic group. We suppose that $\overline{D} \cong \mathbb{C}_{p^\infty}$. Let $\overline{F} = F/G'$ be a p -basic subgroup of \overline{S} . If $\overline{F} \neq \overline{S}$, then $\overline{G}/\overline{F}$ is a direct product of a quasicyclic p -subgroup and an infinite p -divisible abelian subgroup. By Lemma 2.2 of [2] and Lemma 1.1 G is a *BFC*-group.

Assume that $\overline{F} = \overline{S}$. Then by Lemma 26.1 and Proposition 27.1 from [10] $\overline{G}/\overline{F}^p = D^* \times F^*$ is a direct product of a quasicyclic p -subgroup D^* and a p -subgroup F^* of exponent p . Lemma 2.2 of [2] implies $G' = D'S'$. If \overline{F} is not a finitely generated subgroup, then in view of Lemma 1.1 D and G are the *BFC*-groups. Therefore we assume that \overline{F} is a finitely generated subgroup. Since F is a *BFC*-subgroup, $|G' : D'| < \infty$. By Lemma 1.2 D/G'' is a nilpotent group and so D/D' is a Černikov group. This yields that D is a Černikov group. By Lemmas 1.1 and 1.3 D is a *BFC*-group and as a consequence G is the ones.

(2) Now let the divisible part \overline{D} is trivial. If $\overline{F} = \overline{S}$, then the quotient group G/G' is finitely generated or $\overline{G}/\overline{F}^p$ is a direct product of infinitely many cyclic subgroups of order p in which case G is a *BFC*-group. Therefore we assume that $\overline{F} \neq \overline{S}$. If \overline{F} is not finitely generated, in the same manner as above we can prove that G is a *BFC*-group.

Let \overline{F} be a finitely generated subgroup.

(a) Assume that the quotient group $G_1 = \overline{G}/\overline{F}$ is non-torsion. Then there exists a subgroup \overline{F}_0 such that $\overline{F} \leq \overline{F}_0 \leq \overline{G}$ and $\overline{G}/\overline{F}_0$ is torsion-free. As noted in [6] (see also [7, Chapter 2, §6]) $\overline{G}/\overline{F}_0$ contains a subgroup $\overline{T}/\overline{F}_0$ isomorphic to \mathbb{Q}_p . If $\overline{Z}/\overline{F}_0$ is a subgroup of $\overline{T}/\overline{F}_0$ isomorphic to \mathbb{Z} , then $\overline{T}/\overline{Z}$ is a quasicyclic p -group, and it follows that G has a normal *BFC*-subgroup X with $G/X \cong \mathbb{C}_{p^\infty}$. By Lemma 1.2 $G_0 = G/X'F^p$ is a nilpotent group and so by Lemma 26.1 and Proposition 27.1 from [10] $G_0/G'_0 = F_1 \times K_1$ is a direct product of a finite p -subgroup F_1 and an infinite p -divisible abelian subgroup K_1 . Let K_0 be an inverse image of K_1 in G_0 . From what is proved above it follows that K_0 has a normal subgroup K^* with $K_0/K^* \cong \mathbb{C}_{p^\infty}$. If $K^* \neq (K^*)^p$, then Theorem 1.16 of [7] yields that $K_0/(K^*)^p = \overline{X} \times \overline{Y}$ is a direct product of a quasicyclic p -subgroup \overline{X} and some divisible p -subgroup \overline{Y} . Since \overline{Y} is a non-trivial subgroup, it is infinite. Consequently G is a *BFC*-group. Therefore we suppose that $K^* = (K^*)^p$. As above we can prove that K^* contains a G -invariant subgroup L with $K^*/L \cong \mathbb{C}_{p^\infty}$. Hence $K_0/L \cong \mathbb{C}_{p^\infty} \times \mathbb{C}_{p^\infty}$ and so G is a *BFC*-group.

(b) Let $G_1 = \overline{G}/\overline{F}$ be an infinite torsion p' -group. Then without loss of generality we can assume that G_1 is an infinite q -group for some prime q different from p . By B we denote a basic subgroup of G_1 . If $B = G_1$, then the quotient group G/G' is finitely generated or B is an infinitely generated subgroup in which case G is a *BFC*-group.

Let $B \neq G_1$. If B is not a finitely generated subgroup, then Lemma 26.1 and Proposition 27.1 of [10] give that $G_1/B^q \cong \overline{B} \times \mathbb{C}_{q^\infty}$, where \overline{B} is an infinite abelian q -subgroup of exponent q , and this yields that G is a *BFC*-group. Therefore we assume that B is a finitely generated subgroup. Then without loss of generality let $B = 1$ and $G_1 \cong \mathbb{C}_{q^\infty}$. We would like to prove that the commutator subgroup G' is torsion. Since the subgroup G'' is finite, without restricting of generality let $G'' = 1$. But then $\hat{F} = F/\tau(G')$ is an abelian subgroup of $\hat{G} = G/\tau(G')$ and from $G_1 \cong \hat{G}/\hat{F}$ it follows that \hat{G} is an abelian group. This means that G' is a torsion subgroup. By Lemma 1.2 G/F' is a nilpotent group and it has the torsion commutator subgroup. So Corollary 3.3 of [2] yields that G/F' is a torsion group. Hence G is a torsion group and $\overline{G} \cong \mathbb{C}_{q^\infty} \times M$, where M is a finite subgroup, a contradiction with our assumption.

(c) Finally, if $G_1 = \overline{G}/\overline{F}$ is a torsion group and it has a non-trivial p -subgroup, then without loss of generality we can assume that G_1 is a quasicyclic p -group. As in the line (b) this gives that G is a *BFC*-group. The proposition is proved. \square

Lemma 1.5. *Let $G = B\langle x \rangle$ be a product of a normal abelian torsion-free subgroup B and a cyclic subgroup $\langle x \rangle$. If G satisfies $\text{Max-}\overline{BFC}$, then it is either an abelian group or a polycyclic group.*

Proof: If F is any finitely generated subgroup of B , then $\langle F, x \rangle$ is a polycyclic subgroup in G and $\langle F, x \rangle = A\langle x \rangle$ for some G -invariant subgroup A of B . Assume that the quotient group G/A is not finitely generated. Then $A\langle x \rangle$ is a *BFC*-subgroup in view of Lemma 1.1 and consequently it is abelian. Therefore a non-polycyclic group G is abelian, as desired. \square

Lemma 1.6. *If G is a solvable group satisfying $\text{Max-}\overline{BFC}$, then one of the following conditions holds:*

- (i) $G = BU$ is a finitely generated group, where B is a proper torsion normal subgroup of G , U its polycyclic subgroup and $B\langle x \rangle$ is either a *BFC*-subgroup or a finitely generated subgroup for every element x of U ;
- (ii) G is a *BFC*-group;

(iii) $G = DV$ is a product of a normal divisible abelian p -subgroup D and a polycyclic subgroup V .

Proof: Suppose that G is not a BFC -group. Let n be the derived length of G . Then there exists an integer k such that $G^{(k-1)}$ is not a BFC -group, but $G^{(k)}$ is a BFC -group, where $1 \leq k \leq n-1$ and $G^{(0)} = G$. Proposition 1.4 implies that $G^{(k-1)} = G^{(k)}U$ for some polycyclic subgroup U . By Lemma 1.5 $\overline{U} \triangleleft \overline{G}^{(k-1)}$, where $\overline{G}^{(k-1)} = G^{(k-1)}/\tau(G^{(k)}) = \overline{G}^{(k)}\overline{U}$, and so $\overline{U} = \overline{G}^{(k-1)}$. This means that $G^{(k-1)} = \tau(G^{(k)})U$. We denote $\tau(G^{(k)})$ by B .

(a) First we assume that G is not a finitely generated group. Clearly that there is an element u of U such that $H_1 = G^{(k)}\langle u \rangle$ is a non- BFC group. We would like to prove that $H = B\langle u \rangle$ is the ones. Indeed, if H is a BFC -group, then the quotient group $H_1/H'G^{(k+1)}$ is a nilpotent group and by Theorem 2.26 of [10] and Proposition 1.4 it is finitely generated. But then H_1 (and consequently G) is also a finitely generated group, a contradiction. Hence H is a non- BFC group.

(1) Assume that B is an abelian π -subgroup for some set π of primes. If $B = B_1 \times B_2$ is a direct product of an infinite π_1 -subgroup B_1 and an infinite π_2 -subgroup B_2 , where π_1 and π_2 are the disjoint subsets of π such that $\pi = \pi_1 \cup \pi_2$, then it is not difficulty to prove that H is a BFC -group, a contradiction. Thus π is a finite set and $B = P \times S$, where P is an infinite p -subgroup for some prime $p \in \pi$ and S is a finite p' -subgroup. Moreover $P\langle u \rangle$ is a non- BFC group.

(2) If B is not necessary an abelian subgroup, then from the line (1) it follows that B/T is a divisible abelian p -group for some finite H -invariant subgroup T . By Theorem 1.16 of [7] there exists a divisible abelian p -subgroup D of B such that $D \leq Z(B)$ and $B = DT$. Thus $G = DV$, where V is a polycyclic subgroup.

(b) Now let G be a finitely generated group. Then $G = BU$ for some polycyclic subgroup U . Suppose that $B\langle x \rangle$ is not a BFC -group for some $x \in U$. If $B\langle x \rangle$ is not finitely generated, then, as in the line (1) and (2), we can prove that $B\langle x \rangle = D_1V_1$, where D_1 is a normal divisible p -subgroup, V_1 is a polycyclic subgroup and $D_1 \leq B$. By Theorem of [2] $B\langle x \rangle$ contains a proper non- BFC subgroup K . Since $\overline{D_1K} = D_1K/(D_1 \cap K) = \overline{D_1} \rtimes \overline{K}$ and $\overline{D_1}$ is a non-trivial divisible p -subgroup, we conclude that D_1K (and consequently G) contains an infinite properly ascending series of type

$$K < K_1 < \dots < K_n < \dots,$$

a contradiction. This means that $B\langle x \rangle$ is a finitely generated subgroup. The lemma is proved. \square

Example 1.7. If $G = A \rtimes \langle t \rangle$, where $\langle t \rangle$ is an infinite cyclic subgroup, $A \cong \mathbb{C}_{p^\infty}$ and $a^t = a^{1+p}$ ($a \in A$), then G satisfies $\text{Max-}\overline{BFC}$.

If D is a commutative Dedekind domain, A right D -module, $\text{Spec}(D)$ the set of non-trivial prime ideals of D and $P \in \text{Spec}(D)$, then

$$A_P = \{a \in A \mid aP^n = \{0\} \text{ for some positive integer } n = n(a) \in \mathbb{N}\}$$

is said to be the P -component of A , and A is said to be a D -torsion module if

$$A = \{a \in A \mid \text{Ann}(a) \neq \{0\}\}.$$

Lemma 1.8. Let $G = A \rtimes \langle x \rangle$ be a semidirect product of a normal abelian subgroup A of exponent p and an infinite cyclic subgroup $\langle x \rangle$. If G satisfies $\text{Max-}\overline{BFC}$, then it is either a finitely generated group or a *BFC*-group.

Proof: It is clear that A is a right $\mathbb{F}_p\langle x \rangle$ -module with the action determined by the conjugation of x on A . Assume that G is not neither a finitely generated group nor a *BFC*-group. Then A is a $\mathbb{F}_p\langle x \rangle$ -torsion module and by Proposition 2.4 of [8, §8.2]

$$A = \sum_{P \in \text{Spec}(\mathbb{F}_p\langle x \rangle)}^{\oplus} A_P$$

is a module direct sum of its P -component A_P . Without loss of generality we can suppose that $|A : A_Q| < \infty$ for some $Q \in \text{Spec}(\mathbb{F}_p\langle x \rangle)$. Let B be a basic submodule of A_Q . By our hypothesis $B = A_Q$. Since B can be written as a direct product of two infinite G -invariant subgroup of infinite index, we obtain that $B \rtimes \langle x \rangle$ (and consequently G) is a *BFC*-group, a contradiction. The lemma is proved. \square

Proposition 1.9. If G is a non-“finitely generated” non-*BFC* solvable group satisfying $\text{Max-}\overline{BFC}$, then:

- (1) G is a locally nilpotent-by-finite group;
- (2) $G = BU$ is a product of a normal divisible abelian p -subgroup B and a polycyclic subgroup U ;
- (3) $B\langle u \rangle$ is a *BFC*-subgroup for an element $u \in U$ if and only if $u \in C_U(B)$;
- (4) if $B\langle u \rangle$ is a non-*BFC* subgroup for some element $u \in U$, then $[B, \langle u \rangle] = B$;

- (5) if $B\langle u \rangle$ is a non-BFC subgroup for some element $u \in U$, then B is an indecomposable injective $\mathbb{Q}\langle u \rangle$ -module;
- (6) if $B\langle u \rangle$ is a non-BFC subgroup for some $u \in U$, then $A\langle u \rangle$ is a BFC-subgroup for every proper $\mathbb{Z}\langle u \rangle$ -submodule A of B , where the action is induced by the conjugation of u on B ;
- (7) G contains a normal subgroup H of finite index in which every non-BFC subgroup is subnormal;
- (8) G' is a torsion subgroup of G .

Proof: (1) Is obvious.

(2) Follows from Lemma 1.6.

(3) Assume that $H = B\langle u \rangle$ is a BFC-subgroup for some element $u \in U$. If u has a finite order, then $H/H'\langle u \rangle$ is a divisible group and by Theorem 1.16 of [7] $H = Z(H)H'\langle u \rangle$. Consequently $Z(H)$ is a subgroup of finite index in H and H is an abelian group.

Let u be an element of infinite order. Since the subgroup $H'\langle u^s \rangle$ and the quotient group $B\langle u^s \rangle / (H'\langle u^s \rangle)'$ are nilpotent for some integer s , $B\langle u^s \rangle$ is a nilpotent group by Hall theorem [16, Theorem 2.27]. But then $B\langle u^s \rangle$ is an abelian group and therefore as proved above $H/(Z(H) \cap \langle u \rangle)$ is abelian. This yields that H is an abelian group.

(4) If $B\langle u \rangle$ is a non-BFC subgroup for some element u of U and $[B, \langle u \rangle] \neq B$, then $T = [B, \langle u \rangle]\langle u \rangle$ is a BFC-subgroup. Since $B\langle u \rangle/T'$ is a nilpotent group, it is abelian, a contradiction.

(5) It is clear that B is a right $\mathbb{Q}\langle u \rangle$ -module with the action induced by the conjugation of u on B . Furthermore, B is a divisible $\mathbb{Q}\langle u \rangle$ -module and therefore it is injective (see e.g. [11, Theorem 5.28]). By Theorem 2.5 of [15] B has a decomposition as a module direct sum of indecomposable injective $\mathbb{Q}\langle u \rangle$ -submodules. Since $B\langle u \rangle$ satisfies Max- \overline{BFC} , B is an indecomposable module.

(6) Let $B\langle u \rangle$ be a non-BFC group and A a proper submodule of a right $\mathbb{Z}\langle u \rangle$ -module B , where the action is induced by the conjugation of u on B . By F we denote a basic subgroup of A . If $A = F$, then $A\langle u \rangle$ is either a polycyclic group or a BFC-group in view of Lemmas 1.8 and 1.5. Therefore we assume that $F \neq A$. Since B is an indecomposable $\mathbb{Q}\langle u \rangle$ -module, we conclude that F is an infinite group. But then A/A^p is also infinite and so $A\langle u \rangle/A^p$ is a BFC-group by Lemma 1.8. This yields that $A\langle u \rangle$ is the ones.

(7) If V is a nilpotent subgroup of finite index in U and K is any non-*BFC* subgroup of DV , then $D \leq K$. Hence K is a subnormal subgroup of DV .

(8) Is obvious. The proposition is proved. \square

Proof of Theorem 1: (\Rightarrow) Follows from Proposition 1.9.

(\Leftarrow) Suppose that K is a non-*BFC* subgroup of a non-*BFC* group G .

Let G be a group of type (ii) and $\overline{BK} = BK/B'(B \cap K) = \overline{B} \rtimes \overline{K}$. Since BK is a finitely generated subgroup, $\overline{S} = (\overline{B} \cap \overline{S}) \rtimes \overline{K}$, where \overline{S} is a subgroup of \overline{BK} which contains \overline{K} , and \overline{BK} satisfies the maximal condition on normal subgroups by Theorem 5.34 of [10], we conclude that every properly ascending series of type $\overline{K} < \overline{K}_1 < \dots < \overline{K}_n < \dots$ is finite. This means that BK (and consequently G) satisfies $\text{Max-}\overline{BFC}$.

If G is a group of type (iii), then it is clear that $K = (K \cap D)F$, where $F = \langle u_1, \dots, u_t \rangle$ is some finitely generated subgroup. Assume that $K_i = (K \cap D)\langle u_i \rangle$ has the finite commutator subgroup K_i' for all i ($1 \leq i \leq t$). Since the subgroup $\langle K_1', \dots, K_t', F \rangle$ is a finitely generated and $\langle K_1', \dots, K_t', F \rangle = K_0 F$ for some finite F -invariant subgroup $K_0 \leq K \cap D$, $(K/K_0)' = (FK_0/K_0)'$ is a finite subgroup and therefore K is a *BFC*-subgroup, a contradiction. Hence $(K \cap D)\langle u \rangle$ is non-*BFC* subgroup for some $u \in F$ and by our hypothesis $D = K \cap D \leq K$. The theorem is proved. \square

Corollary 1.10. *A solvable group G satisfies $\text{Max-}\overline{BFC}$ if and only if it is of one of the following types:*

- (i) G is a *BFC*-group;
- (ii) $G = BU$ is a finitely generated group, where B is a proper torsion normal subgroup of G , U its polycyclic subgroup and $B\langle x \rangle$ is either a *BFC*-subgroup or a finitely generated subgroup for every element x of U ;
- (iii) $G = DU$ is a product of a normal divisible abelian p -subgroup D and a polycyclic subgroup U with $D \leq \bigcap \{H \mid H \text{ is a non-}\overline{BFC} \text{ subgroup of } G\}$.

2. Groups with $\text{Min-}\overline{BFC}$

In this section we prove that a group which have a descending series with abelian or finite factors and satisfying $\text{Min-}\overline{BFC}$ is either a *BFC*-group or a Černikov group.

Lemma 2.1. *If G is a non-perfect group in which every proper normal subgroup is a *BFC*-subgroup, then G is a *BFC*-group or $G = G'\langle x \rangle$, where $x^{p^n} \in G'$ for some prime p and some positive integer n .*

Proof: By Theorem 21.3 of [10] $\overline{G} = G/G' = \overline{D} \times \overline{S}$ is a direct product of the divisible part \overline{D} and a reducible subgroup \overline{S} . Let \overline{B} be a p -basic subgroup of \overline{S} . If \overline{B} is not a finitely generated subgroup, then by Lemma 26.1 and Proposition 27.1 of [10] $\overline{S}/\overline{B}^p = B_1 \times S_1$ is a direct product of an infinite abelian subgroup B_1 of exponent p and a p -divisible subgroup S_1 . By Corollary 2.1 of [2] G is a BFC -group.

Let \overline{B} be a finitely generated subgroup. If \overline{D} is a non-trivial subgroup or $\overline{B} = \overline{S}$, then Lemma 26.1, Proposition 27.1 of [10] and Lemma 2.2 of [2] yield that G is a BFC -group. Finally, from $\overline{B} = \overline{S}$ and $\overline{D} = \overline{1}$ in view of Corollary 2.1 of [2] it follows that G is a BFC -group or G/G' is a cyclic p -group for some prime p , as desired. \square

Proof of Theorem 2: Assume that G is neither a BFC -group nor a Černikov group. Since G satisfies $\text{Min-}\overline{BFC}$, we may invoke [9, Theorem 2.2] and obtain in this way that G is an FC -group. Choose

$$G = G_0 \geq G_1 \geq \cdots \geq G_n$$

such that every G_i is not a BFC -subgroup ($i = 0, \dots, k-1$), while every proper normal subgroup of G_n is a BFC -group. Since G has a descending series whose factors are finite or abelian, there exists a normal subgroup N in G_n such that G_n/N is finite or abelian. From Lemma 2.1, the quotient G_n/N is finite in both cases. Hence there exists a finite subset F of G_n such that $G_n = NF$, and every element in G_n is of the form hf for suitable $h \in N$, $f \in F$. However, since G is an FC -group, every $f \in F$ has just finitely many conjugates in G_n . And for $h \in N$ the number of conjugates of h in G_n is bounded by $|G_n : N||N : C_N(h)|$ (note that N is a BFC -group). Hence G_n itself becomes a BFC -group. This contradiction shows that G must be a BFC -group or a Černikov group. \square

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